Quantized Elastica as a Loop Space

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July 5, 2014, Kansai Univ.

This talk is based on the works with Yoshihiro Ônishi, and Emma Previato

I work in Canon Inc. Dep. of Engineering Math. as Applied Mathematician and Applied Physicist,

whereas I study Pure Math as a private researcher.

1992 Phys.Rev.A: M-Tsuru, Physical relation between quantum mechanics and soliton on a thin elastic rod,
1995 Mod.Phys.Lett.A: M, The relation of Lemniscate and a loop soliton as 3/2 and 1 pin fields along the modified Korteweg-de Vries equation
1998 J.Phys.A: M, Statistical Mechanics of elastica on plane,
1999 J.Geom.Phys.: M, Statistical mechanics of non-stretching elastica in three dimensional space,
2002 J.Geom.Phys.: M, Hyperelliptic loop solitons with genus g: investigation of a quantized elastica,
2003 Rev.Math.Phys.: M-Ônishi, On the moduli of a quantized elastica in P and KdV flows: study of hyperelliptic curves as an extension of Eulers perspective of elastica I,
2007 J. Diff. Eqns.: M, Reality conditions of loop solitons genus g: hyperelliptic am functions Electron,
2008 Crelle′s J: Eilbeck-Enolskii-M-Ônishi-Previato, Addition formulae over the Jacobian pre-image of hyperelliptic Wirtinger varieties
2008 J.Phys.A: M, Relations in a quantized elastica
2008, 2014, J.Math.Soc.Japan: M-Previato, Jacobi inversion on strata of the Jacobian of the C_{rs} curve $y^r = f(x)$. I, II
2009 Israel J. Math: M-Previato, A generalized Kiepert formula,
2010 J. Geom. Symm. Phys: M, Eulers Elastica and Beyond,
2013 Annales de l Institut Fourier: Kodama-M-Previato, Quasi-periodic and periodic solutions of the Toda lattice **+60 papers**

Quantized Elastica as a Loop Space

Shigeki Matsutani (Sagamihara)

July 7, 2014, Kansai Univ.

Menu

1. History of Elastica

**(Origin of Variational Principle,
Differential Geometry, Algebraic Geometry,
Elliptic function, Moduli of Elliptic Curves)**

History of Elastica I

Preliminary

Immersion:

$Z : S^1 \hookrightarrow \mathbb{C}$ smooth ($|\partial_s Z| = 1$).

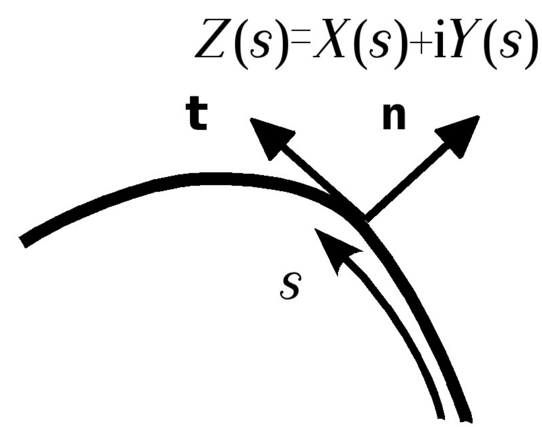
$Z(s) = X(s) + \sqrt{-1}Y(s)$,

$\mathbf{t} = \partial_s Z = \mathrm{e}^{\sqrt{-1}\phi}$, $\phi \in \mathcal{C}^\infty(\kappa^{-1}S^1, \mathbb{R})$

$= \cos\phi + \sqrt{-1}\sin\phi$

$\mathbf{n} = \sqrt{-1}\mathbf{t} = \sqrt{-1}\partial_s Z.$

$\kappa : \mathbb{R} \to S^1$



Curvature & Frenet-Serret relation

$$\mathbf{t} := \partial_s Z, \quad \partial_s \mathbf{t} = k\mathbf{n}, \quad \partial_s \mathbf{n} = -k\mathbf{t}, \quad (\partial_s^2 Z = \sqrt{-1}k\partial_s Z), \qquad (1)$$

$k := \partial_s \phi$: the curvature; $k = 1/$ curvature radius.

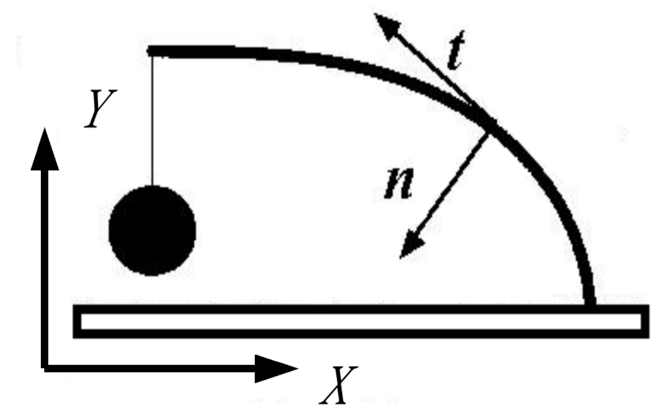
History of Elastica II

What is elastica?

Elastica is an elastic curve or an ideal thin non-stretching elastic beam.

the model of a bent paper, a bent rod, wire, rope and so on.

The elastica problem was proposed by James (Jacob) Bernoulli (1654-1705) in 1691:
''What shape does an elastica have in a plane?''



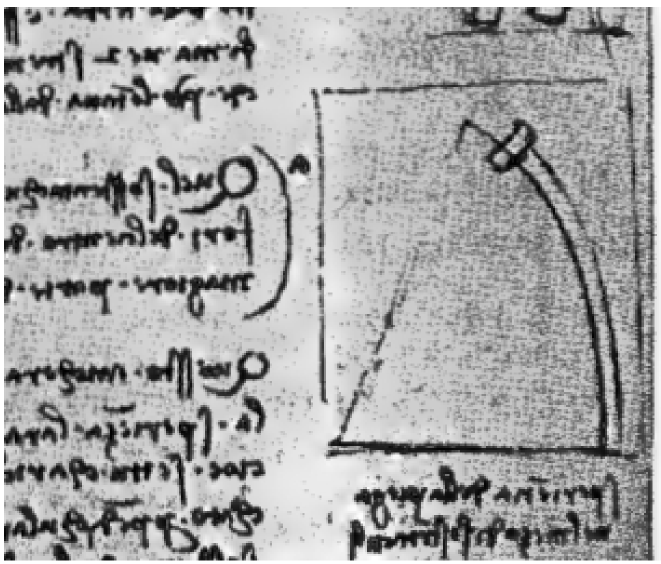
History of Elastica III

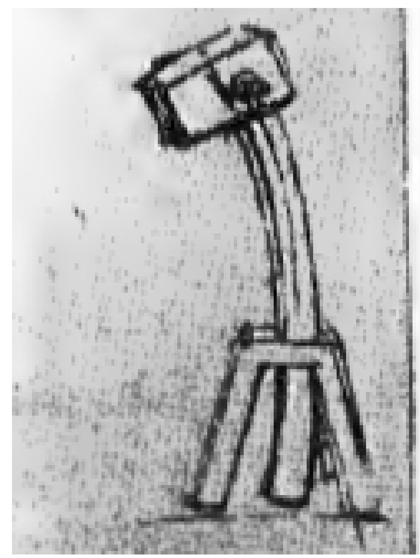
Origin of elastica

Leonardo da Vinci (1452-1519)

History of Elastica IV
Origin of elastica

Leonardo da Vinci (1452-1519) drew the pictures of bent beams





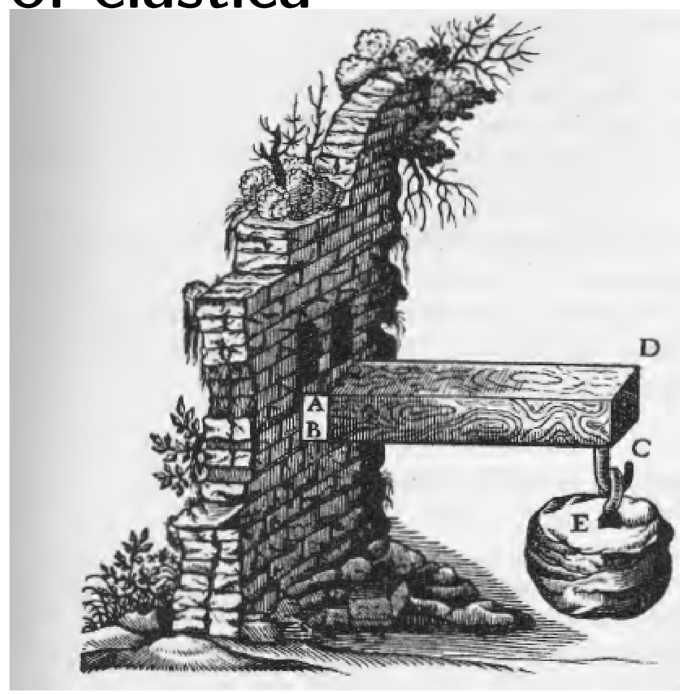
History of Elastica V

Origin of elastica

Galileo Galilei (1564-1654)

History of Elastica VI

Origin of elastica



Galileo Galilei (1564-1654) investigated bent beams.

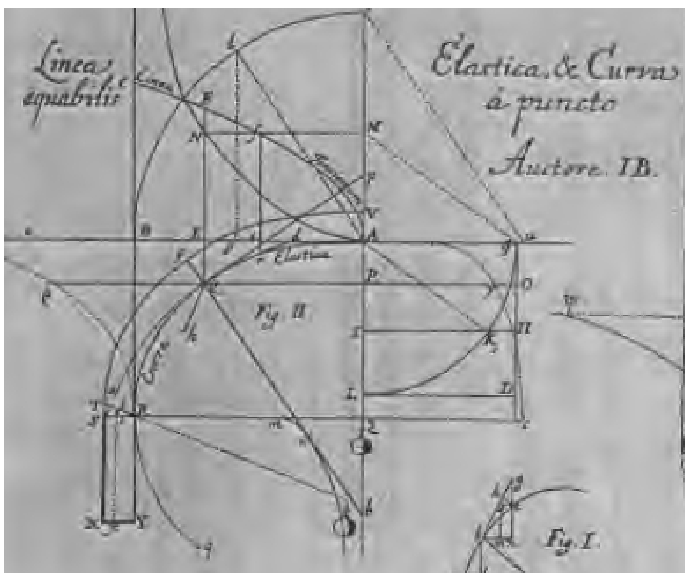
It is a problem of cantilever.

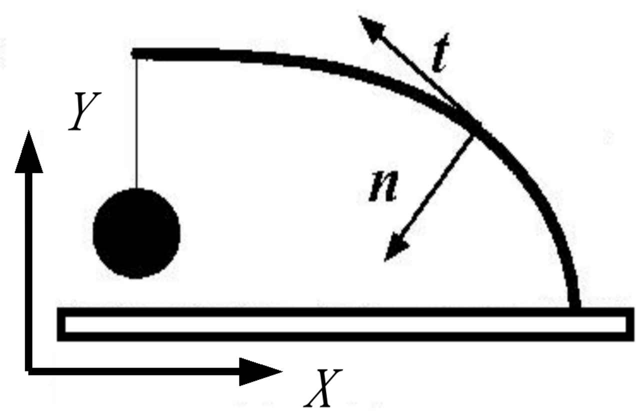
History of Elastica VII
Origin of elastica

James Bernoulli (1654-1705)

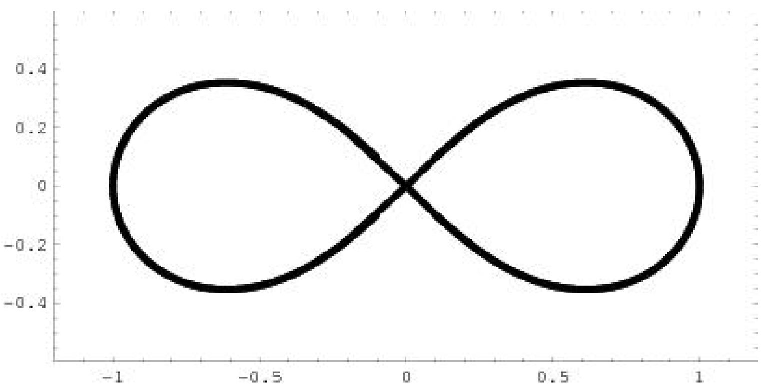
History of Elastica VIII
Origin of elastica

James Bernoulli (1654-1705) proposed the Elastica problem and found the fact that the elastic force is proportional to k and the Lemniscate integral: $s = \int_X^1 \frac{dX}{\sqrt{1-X^4}}$.





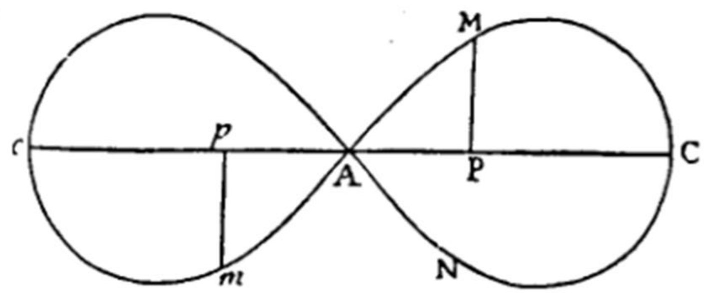
History of Elastica IX
Lemniscate and Elastica
James Bernoulli defined the Lemniscate curve of eight figure.



Lemniscate

$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$

ϕ_{lemni}**:tangential angle**



3c. Class 5

Elastica of Eight-Figure

ϕ_{elas}**:tangential angle**

$$\phi_{\text{lemni}} = \frac{3}{2}\phi_{\text{elas}} \quad \text{[M 1995]}$$

History of Elastica X

Origin of elastica

Daniel Bernoulli (1700-1782)

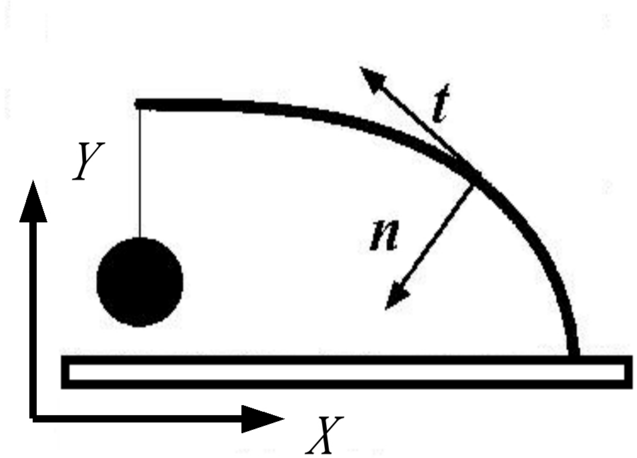
History of Elastica XI

Classical Elastica Problem

Daniel Bernoulli (1700-1782) discovered **the least principle** 1738 in a letter to Euler (1707-1783).

An elastica is realized as the least point of the energy,

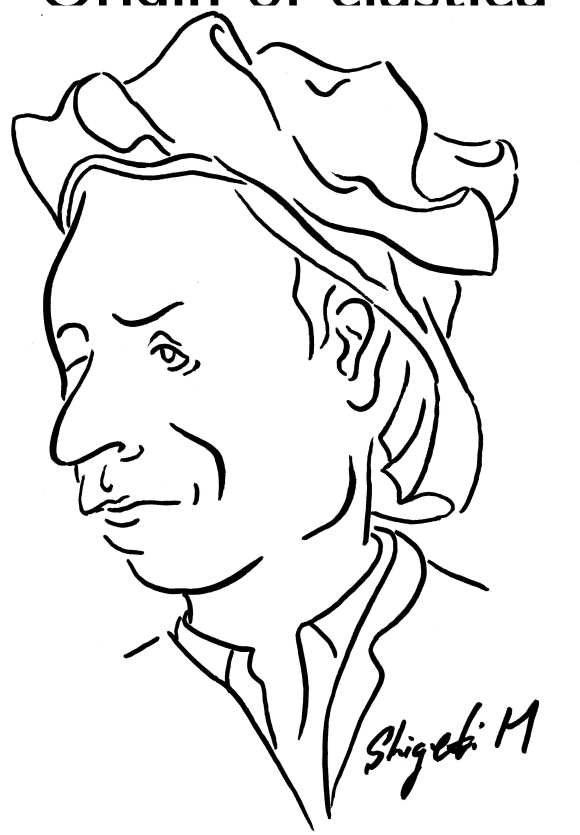
$$\int_{S^1} ds\ k^2(s) = \int_{S^1} ds\ (\partial_s \phi(s))^2$$
$$= \int_{S^1} g^{-1} dg * g^{-1} dg, \quad g \in \mathrm{U}(1)$$



Elastica problem is the oldest harmonic map problem.

History of Elastica XII

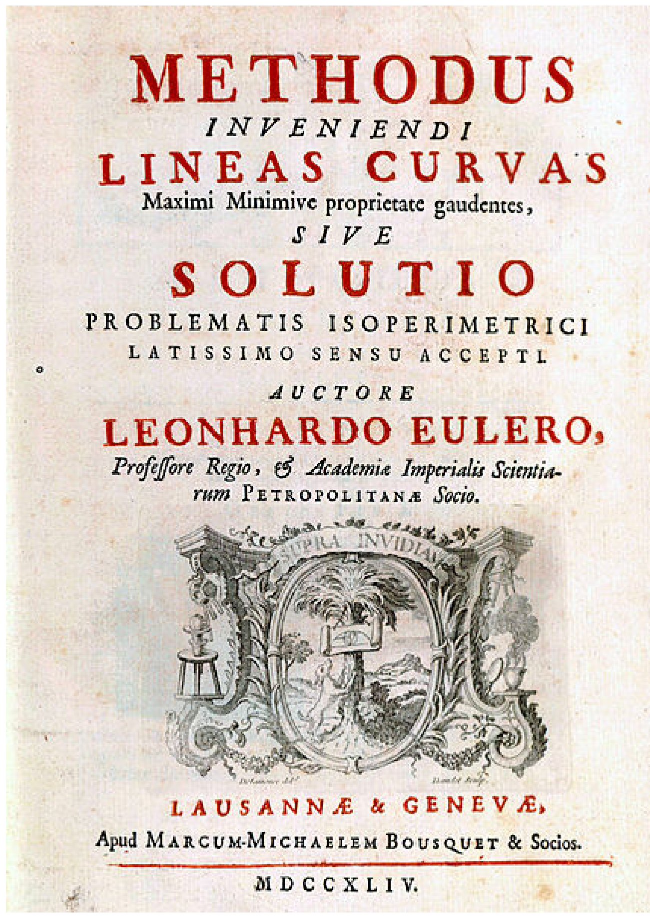
Origin of elastica



Euler (1707-1783)

History of Elastica XIII

“Methodus inveniendi lineas curvas” 1744 ”Method”



METHODUS
INVENIENDI
LINEAS CURVAS
Maximi Minimive proprietate gaudentes,
SIVE
SOLUTIO
PROBLEMATIS ISOPERIMETRICI
LATISSIMO SENSU ACCEPTI.
AUCTORE
LEONHARDO EULERO,
Professore Regio, & Academiæ Imperialis Scientiarum PETROPOLITANÆ *Socio.*

SUPRA INVIDIAM

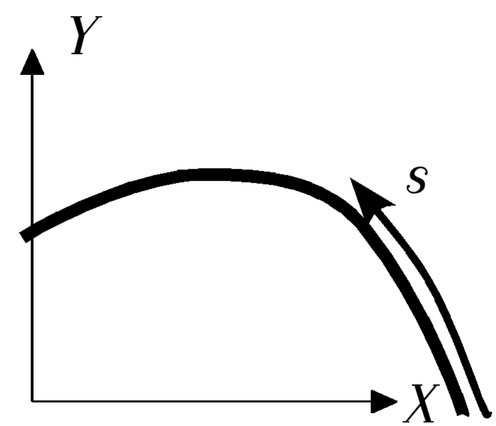
LAUSANNÆ & GENEVÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.
MDCCXLIV.

History of Elastica XIV

Euler's Classical Results of Elastica : 1744 "Method"

$$s = \int^X \frac{\lambda^2 dX}{\sqrt{\lambda^4 - (\alpha + \beta X + \gamma X^2)^2}},$$

$$Y = \int^X \frac{(\alpha + \beta X + \gamma X^2) dX}{\sqrt{\lambda^4 - (\alpha + \beta X + \gamma X^2)^2}}.$$



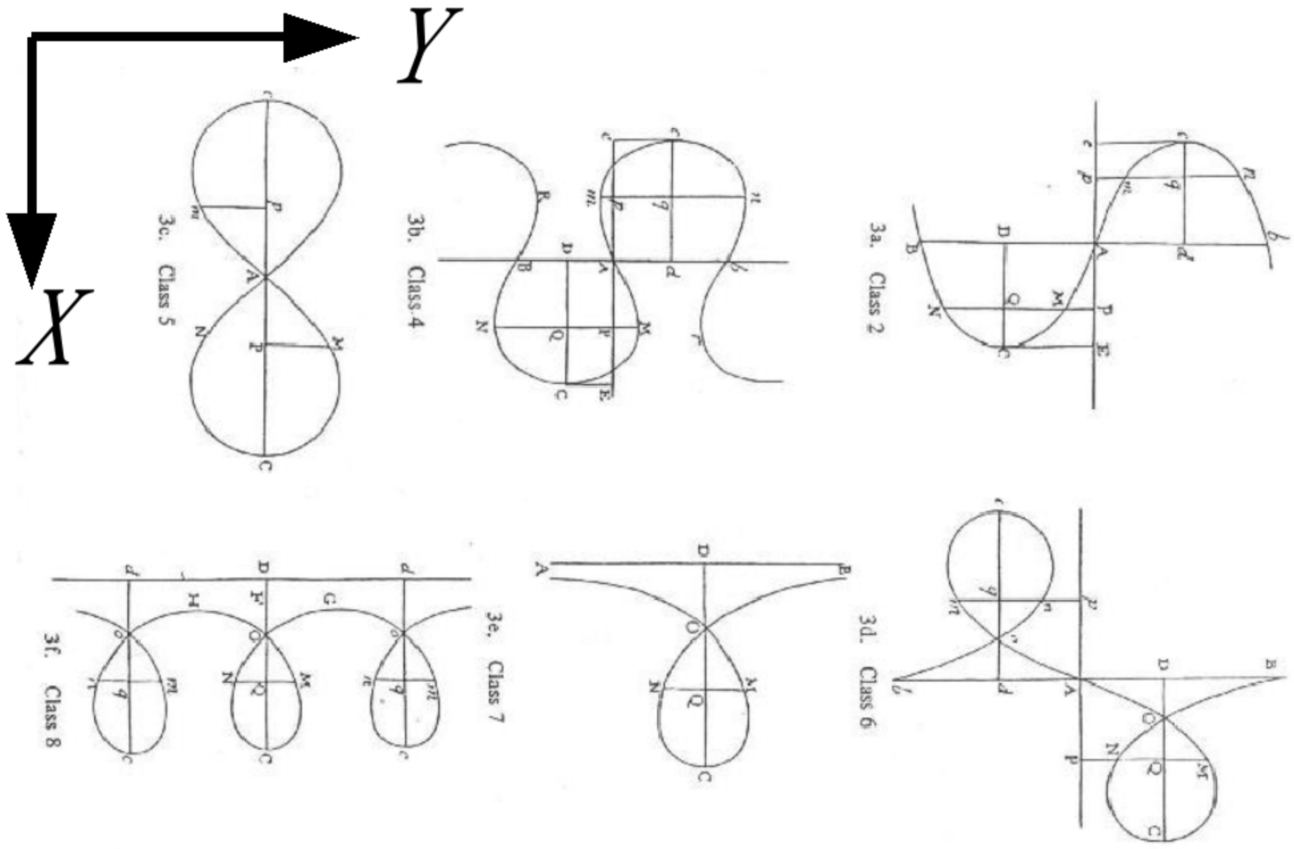
Euler relation (M-Previato 2014)

$$X(s) - X_0 \;=\; \frac{1}{4}k(s)$$

affine coordinate $\propto$ affine connection

History of Elastica XV

Euler's Classical Results of Elastica (1744)



History of Elastica XVI

Euler's Classical Results of Elastica (1744)

It is related to

1. Harmonic Map $S^1 \to$U(1),
2. Variational Method
3. Curve as Differential Geometry
4. Curve as Algebraic Geometry
5. Elliptic Function
6. Moduli of Elliptic Curves

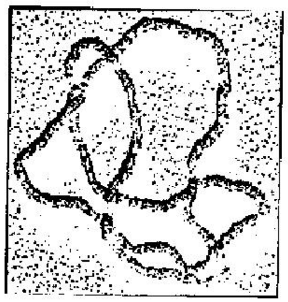
2. Motivation

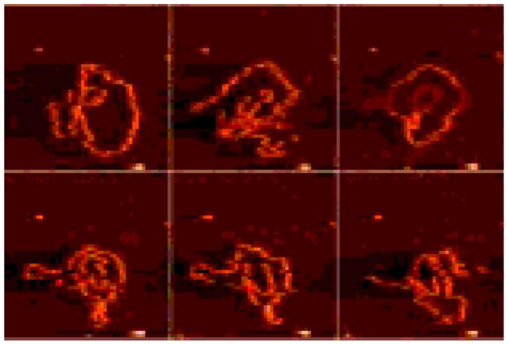
Quantization of Elastica
(Statistical Mechanics of Elasticas)

quantization of geometry

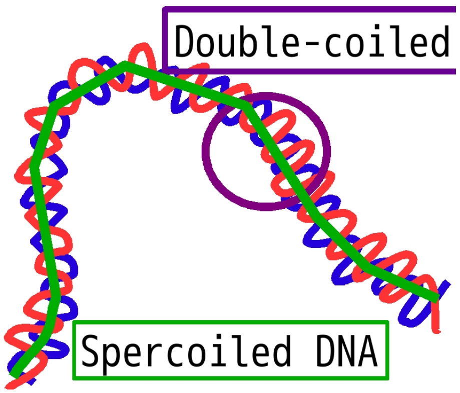
Motivation I: **Quantization of Elastica**

Pictures of DNAs by atomic force microscopes shows the supercoils.





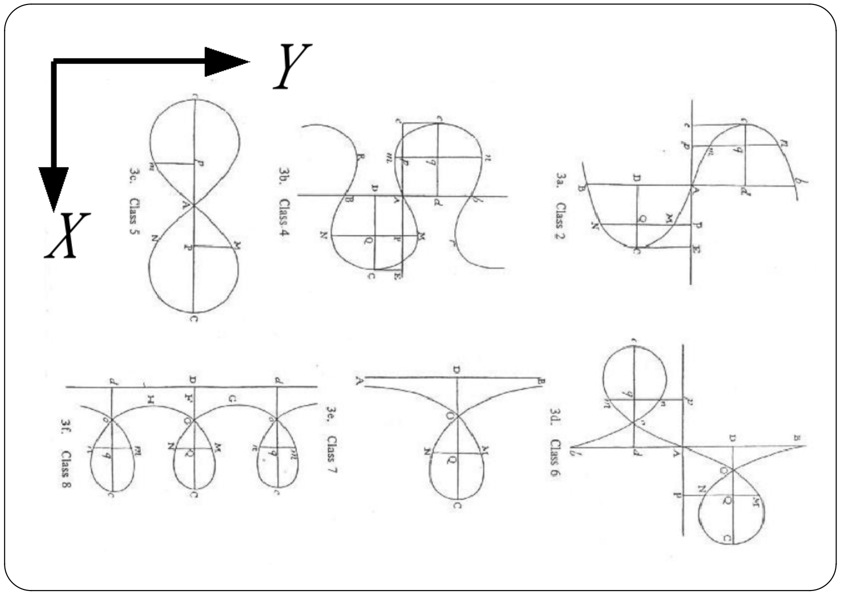
Pictures of DNAs by atomic force microscopes

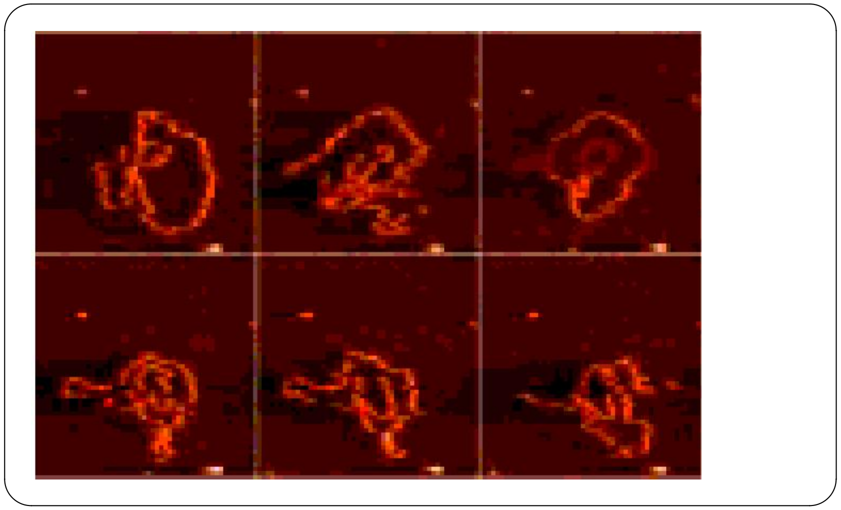


These shapes are super-coils rather than double coils.
Super-coil is weakly governed by the elastic force!

Motivation II: **Quantization of Elastica**

Too simple | **Complicate**





→ Why the shapes of super-coiled DNA due to elastic force are complicate?

Motivation III: **Quantization of Elastica**

The reasons why the shapes of super-coiled DNA are complicate are several ones: the chemical effects, stretching effects, the solvent effects, **the heat effects** and so on.

$\longmapsto$ **I have been studying a statistical mechanics of elastica, which I call quantized elastica due to** $\hbar \leftrightarrow \sqrt{-1}/\beta$**.**

In order to understand the quantization of geometrical objects, we consider the quantized elastica.

Motivation IV:
Physical Motivation: Partition Function of Elasticas

The **partition function** of the quantized elastica at the temperature $T := 1/\beta$, is **formally** given by

$$\mathcal{W}_{\mathrm{elas}}[\beta] = \int_{\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}} DZ \exp(-\beta\mathcal{E}[Z]),$$

with energy for the curvature k

$$\mathcal{E}[Z] := \oint ds k^2,$$

for its domain

$$\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}} := \{Z : S^1 \to \mathbb{C} | \oint dZ = 2\pi,\ |\partial_s Z| = 1\}/\sim,$$

where $\sim$ means the euclidean moves, and trivial coordinate transformation: $g_{s_0} Z(s) = Z(s - s_0)$ for $g_{s_0} \in \mathrm{U}(1)$.

Motivation V :

Mathematical Motivation: Moduli of Quantized Elastica

It is a mathematical problem how we classify the **moduli of isometric loops**,

$$\mathcal{M}^{\mathbb{C}}_{\text{elas}} = \{Z : S^1 \to \mathbb{C} | \oint dZ = 2\pi,\ |\partial_s Z| = 1\}/ \sim,$$

from viewpoints of the energy,

$$\mathcal{E}[Z] := \oint ds k^2.$$

This is a loop space in category of differential geometry.

Motivation VI :
Mathematical Motivation: Moduli of Quantized Elastica

It is a mathematical problem how we evaluate the **moduli of isometric loops with isoenergy** E,

$$\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},E} = \{Z \in \mathcal{M}^{\mathbb{C}}_{\mathrm{elas}} \mid \mathcal{E}[Z] = E\}, \quad \mathcal{M}^{\mathbb{C}}_{\mathrm{elas}} = \coprod_E \mathcal{M}^{\mathbb{C}}_{\mathrm{elas},E},$$

in order to evaluate

$$\mathcal{W}_{\mathrm{elas}}[\beta] = \int_0^\infty dE \ \mathrm{Vol}(\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},E}) \exp(-\beta E).$$

This is an isometric and isoenergy loop space.

Since wild curves have higher energy than the others, we can assume that the map Z is **analytic**.

Motivation VII :

Mathematical Motivation: Moduli of Quantized Elastica

We wish to investigate $\mathcal{M}^{\mathbb{C}}_{\text{elas}}$, $\mathcal{M}^{\mathbb{C}}_{\text{elas},E}$ and $\mathcal{M}^{\mathbb{C}}_{\text{elas},\partial_s Z}$.

$$\mathcal{M}^{\mathbb{C}}_{\text{elas}} = \{Z : S^1 \to \mathbb{C} : \text{analytic} \mid \oint dZ = 2\pi,\ |\partial_s Z| = 1\}/\sim,$$
$$\mathcal{M}^{\mathbb{C}}_{\text{elas},E} = \{Z \in \mathcal{M}^{\mathbb{C}}_{\text{elas}} \mid \mathcal{E}[Z] = E\},$$
$$\mathcal{M}^{\mathbb{C}}_{\text{elas},\text{SO}(2)} = \{\partial_s Z \mid Z \in \mathcal{M}^{\mathbb{C}}_{\text{elas}}\}.$$

$$\mathcal{M}^{\mathbb{C}}_{\text{elas},\text{SO}(2)} \subset \mathcal{M}_{\text{SO}(2)} := \{f : S^1 \to \text{SO}(2) : \text{analytic }\}.$$

SO(2) trivially acts on $\mathcal{M}^{\mathbb{C}}_{\text{elas}}$ or a stabilizer, i.e.,

$$g_\theta Z = Z \in \mathcal{M}^{\mathbb{C}}_{\text{elas}}, \quad (g_\theta Z(s) = Z(s-\theta)).$$

3. Infinitesimal Isometric Diffeo. $\mathfrak{idiff}$

Isometric Diffeomorphism (IDIFF)

$\equiv$ Arc-length preserving Diffeomorphism

$\equiv$ Non-stretching deformation (Curve flow)

Infinitesimal Isometric Diffeomorphism ($\mathfrak{idiff} = d\mathrm{IDIFF}|_e$)

Infinitesimal Isometric Diffeo. $\mathfrak{idiff}$ II:
Complex analysis viewpoint I

1st: tangential angle $\qquad \phi = \dfrac{1}{\sqrt{-1}} \log Z'(s_0)$

2nd: curvature $\qquad k := \dfrac{1}{\sqrt{-1}} \dfrac{Z''(s_0)}{Z'(s_0)} = \partial_s \phi$

3rd: Schwarz derivative

$$\{Z, s_0\}_{\mathrm{SD}} = \left[\frac{Z'''(s_0)}{Z'(s_0)} - \frac{3}{2}\left(\frac{Z''(s_0)}{Z'(s_0)}\right)^2\right] = \left(\sqrt{-1}\partial_s k - \frac{1}{2}k^2\right).$$

Euler-Bernoulli energy function is given by

$$\mathcal{E}[Z] := -2\oint \{Z, s\}_{\mathrm{SD}} ds = \oint k^2 ds.$$

Infinitesimal Isometric Diffeo. $\mathfrak{idiff}$ I: **Complex analysis viewpoint II**

Complex analysis viewpoint shows:

Proposition Tjurin (1974) ($s_1 < s_0 < s_2$)

$$\frac{1}{2}\log\frac{Z(s_2)-Z(s_1)}{s_2-s_1} = \frac{1}{2}\boxed{\log Z'(s_0)} + \frac{1}{2}\frac{1}{2!}\boxed{\frac{Z''(s_0)}{Z'(s_0)}}(s_1+s_2)$$
$$+\frac{1}{2}\frac{1}{3!}\left[\frac{Z'''(s_0)}{Z'(s_0)} - \frac{3}{4}\left(\frac{Z''(s_0)}{Z'(s_0)}\right)^2\right](s_2^2+s_1^2)$$
$$+\frac{1}{2}\frac{1}{3!}\boxed{\left[\frac{Z'''(s_0)}{Z'(s_0)} - \frac{3}{2}\left(\frac{Z''(s_0)}{Z'(s_0)}\right)^2\right]}(s_1 s_2) + \cdots,$$

The left hand side appears in the Lagrange inversion formula, Replicable function, dKP theory and so on.

Infinitesimal Isometric Diffeo. $\mathfrak{idiff}$ III:
Infinitesimal Isometric Deformation I

1. Let us consider the fiber $T_Z\mathcal{M}^{\mathbb{C}}_{\text{elas}}$ of the tangent space $T\mathcal{M}^{\mathbb{C}}_{\text{elas}}$ at $Z \in \mathcal{M}^{\mathbb{C}}_{\text{elas}}$, or the infinitesimal flow of $t \in (-\varepsilon, \varepsilon)$ in $\mathcal{M}^{\mathbb{C}}_{\text{elas}}$ at Z;

$$t : (-\epsilon, \epsilon) \to \mathcal{M}^{\mathbb{C}}_{\text{elas}}, \quad \text{and} \quad \text{consider } \partial_t Z|_{t=0}.$$

2. There is a trivial flow fixing a point $Z \in \mathcal{M}^{\mathbb{C}}_{\text{elas}}$,

$$\partial_t Z = c\partial_s Z, \quad c \in \mathbb{R}, \quad \text{or} \quad Z(s) = Z(s + ct).$$

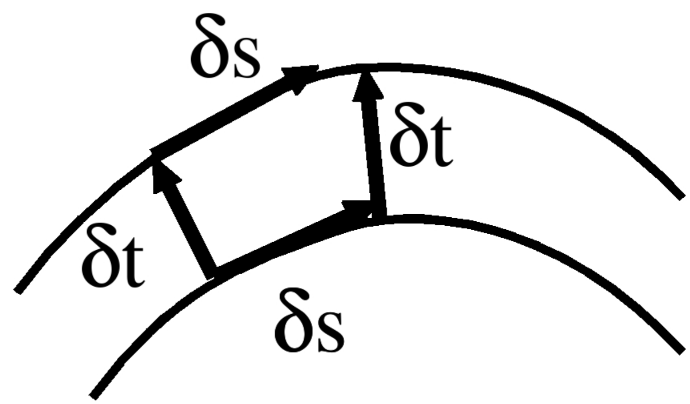
Infinitesimal Isometric Diffeo. 𝔦𝔡𝔦𝔣𝔣 IV:
Infinitesimal Isometric Deformation II

3. For the isometric (non-stretching, arc-length preserving) deformation parameter $t \in (-\varepsilon, \varepsilon)$, the non-stretching condition is

$$[\partial_t, \partial_s] Z = 0$$

4. The non-stretching condition $[\partial_t, \partial_s] Z = 0$ means

$$\partial_s \partial_t \phi = \partial_t \partial_s \phi = \partial_t k.$$



Infinitesimal Isometric Diffeo. $\mathfrak{idiff}$ V:
Infinitesimal Isometric Deformation III

4. By letting

$$\mathcal{A}^0_{S^1}(\mathbb{R}) := \mathcal{C}^\omega(S^1, \mathbb{R}), \quad \mathcal{A}^0_{S^1}(\mathbb{C}) := \mathcal{C}^\omega(S^1, \mathbb{C}),$$

and

$$\mathcal{A}^1_{S^1}(\mathbb{R}) : \mathcal{C}^\omega(S^1, \mathbb{R})\text{-valued one form of } S^1,$$

we consider the non-stretching condition of

$$\partial_t Z(s) = U(s)\partial_s Z = (U_r(s) + \sqrt{-1}U_i(s))\partial_s Z$$

for

$$U = U_r + \sqrt{-1}U_i \in \mathcal{A}^0_{S^1}(\mathbb{C}) \text{ and } U_r, U_i \in \mathcal{A}^0_{S^1}(\mathbb{R}).$$

Infinitesimal Isometric Diffeo. $\mathfrak{i}\mathfrak{d}\mathfrak{i}\mathfrak{f}\mathfrak{f}$ VI:
Infinitesimal Isometric Deformation IV

6. The non-stretching condition $[\partial_t, \partial_s]Z = 0$ leads the following relations **(Goldstein-Petrich 1991)**;

$$\begin{aligned}
\partial_t(\partial_s Z) &= \partial_t \mathrm{e}^{\sqrt{-1}\phi} = \sqrt{-1}\partial_t\phi\ \partial_s Z,\\
\partial_s(\partial_t Z) &= \partial_s[(U_r + \sqrt{-1}U_i)\partial_s Z]\\
&= [(\partial_s U_r - kU_i) + \sqrt{-1}(\partial_s U_i + kU_r)]\ \partial_s Z.
\end{aligned}$$

$$\mapsto \quad \boxed{(\partial_s U_r - kU_i) = 0, \quad \text{and} \quad \partial_t\phi = (\partial_s U_i + kU_r),}$$

$$\mapsto \quad \boxed{\begin{aligned}
\partial_s U_r &= kU_i, & \text{and}\quad \partial_t k &= \partial_s(\partial_s U_i + k\partial_s^{-1}kU_i)\\
U_r &= \partial_s^{-1}kU_i, & &= (\partial_s^2 + \partial_s k \partial_s^{-1}k)U_i.
\end{aligned}}$$

Infinitesimal Isometric Diffeo. $\mathfrak{idiff}$ VII:

Infinitesimal Isometric Deformation V

7. $\partial_s U_r = k U_i$ ($U_r = \partial_s^{-1} k U_i$) means the $\mathfrak{so}(2)$-condition,

$$\partial_s \begin{pmatrix} U_r \\ U_i \end{pmatrix} = \begin{pmatrix} 0 & k \\ \partial_s k^{-1} \partial_s & 0 \end{pmatrix}$$

8. $\partial_s U_r = k U_i$ ($\partial_s U_r ds = k U_i ds$) also means the injection,

$$\begin{aligned} \ell_d : \mathcal{A}^0_{S^1}(\mathbb{R}) &\to d\mathcal{A}^0_{S^1}(\mathbb{R}) \subset \mathcal{A}^1_{S^1}(\mathbb{R}), \\ U_i &\mapsto dU_r = \partial_s U_r ds = k U_i ds = \ell_d(U_i). \end{aligned}$$

Infinitesimal Isometric Diffeo. $\mathfrak{idiff}$ VIII: **Infinitesimal Isometric Deformation VI**

9. Note that for $c \in \mathbb{R}$, $\partial_t Z = c\partial_s Z$ means a trivial flow in $\mathcal{M}^{\mathbb{C}}_{\text{elas}}$, and thus ℓ_d $(dU_r = \ell_d(U_i))$ induces

$$\ell_r : U_i \mapsto U_r, \quad \ell_r : \mathcal{A}^0_{S^1}(\mathbb{R}) \to \mathcal{A}^0_{S^1}(\mathbb{R}),$$

up to the constant translation, and thus

$$\left(\ell_r(U_i) = a + \int^s \ell_d(U_i) ds \text{ for } a \in \mathbb{R} \text{ such that } \oint \ell_r(U_i) ds = 0\right)$$

Infinitesimal Isometric Diffeo. $\mathfrak{idiff}$ IX:
Infinitesimal Isometric Deformation VII

Proposition (Brylinski 1993)

The fiber $T_Z\mathcal{M}^{\mathbb{C}}_{\text{elas}}$ at $Z \in \mathcal{M}^{\mathbb{C}}_{\text{elas}}$ is bijection to $\mathcal{A}^0_{S^1}(\mathbb{R})$ by

$$T_Z\mathcal{M}^{\mathbb{C}}_{\text{elas}} \approx \mathcal{A}^0_{S^1}(\mathbb{R}), \quad (\partial_t Z = \ell(U_i)\partial_s Z,\ U_i \in \mathcal{A}^0_{S^1}(\mathbb{R})),$$

where $\ell : \mathcal{A}^0_{S^1}(\mathbb{R}) \to \mathcal{A}^0_{S^1}(\mathbb{C})$, $(\ell := \ell_r + \sqrt{-1}\text{id})$, for $U_i \in \mathcal{A}^0_{S^1}(\mathbb{R})$).

The tangential space is determined by the velocity U_i for the normal direction.

Note that $\phi \notin \mathcal{A}^0_{S^1}(\mathbb{R})$.

Infinitesimal Isometric Diffeo. $\mathfrak{idiff}$ X:
Infinitesimal Isometric Deformation VIII

Proposition

The curvature k in $T_Z\mathcal{M}^{\mathbb{C}}_{\text{elas}}$ at $Z \in \mathcal{M}^{\mathbb{C}}_{\text{elas}}$ is given by

$$\partial_t k = \Omega_i U_i \quad U_i \in \mathcal{A}^0_{S^1}(\mathbb{R}),$$

where

$$\Omega_i := \partial_s(k\partial_s^{-1}k + \partial_s).$$

The curvature k is given by

$$k ds = d\log g = g^{-1}dg, \quad g \in \mathrm{SO}(2).$$

4. Infinitesimal Isometric & Isoenergy Diffeo., $\mathfrak{iidiff}$

For $Z \in \mathcal{M}^{\mathbb{C}}_{\text{elas},E} \subset \mathcal{M}^{\mathbb{C}}_{\text{elas}}$,

consider $T_Z\mathcal{M}^{\mathbb{C}}_{\text{elas},E} \subset T_Z\mathcal{M}^{\mathbb{C}}_{\text{elas}}$.

Infinitesimal Isometric & Isoenergy Diffeo. $\mathfrak{iidiff}$ I:

Lemma

For $U_i \in \mathcal{A}^0_{S^1}(\mathbb{R})$, the deformation $\partial_t Z = \ell(U_i)\partial_s Z$ or $\partial_t k = \partial_s(k\partial_s^{-1}k + \partial_s)U_i$ is not isoenergy in general.

Proof:

$$\begin{aligned}
\partial_t \mathcal{E} &= \partial_t \oint k^2 ds = 2\partial_t \oint k\partial_t k ds = \oint k\partial_s(k\partial_s^{-1}k + \partial_s)U_i ds \\
&= 2\oint k\partial_s(k\partial_s^{-1}kU_i) + 2\oint k\partial_s^2 U_i ds \\
&= -2\oint(\partial_s k)k\partial_s^{-1}kU_i ds + 2\oint(\partial_s^2 k)U_i ds \\
&= -\oint(\partial_s(k^2))\partial_s^{-1}kU_i ds + 2\oint(\partial_s^2 k)U_i ds \\
&= \oint k^2 kU_i ds + 2\oint(\partial_s^2 k)U_i ds = \oint(k^3 + 2\partial_s^2 k)U_i ds. \quad \blacksquare
\end{aligned}$$

Infinitesimal Isometric & Isoenergy Diffeo. $\mathfrak{iidiff}$ II:

The related loop group of $\mathcal{M}^{\mathbb{C}}_{\text{elas}}$:

$$\mathcal{M}^{\mathbb{C}}_{\text{elas,SO(2)}} = \{\partial_s Z \mid Z \in \mathcal{M}^{\mathbb{C}}_{\text{elas},S^1}\} \subset \mathcal{M}_{\text{SO(2)}}.$$

Key Fact 1: Trivial deformation

1. There is a **trivial** isometric & isoenergy diffeomorphism,

$$\partial_t Z = \partial_s Z \quad \text{with} \quad \partial_t k = \partial_s k, \quad \mathfrak{so}(2)$$

Infinitesimal Isometric & Isoenergy Diffeo. $\mathfrak{iidiff}$ III:

Lemma

For $\mathcal{A}^0_{S^1,c}(\mathbb{R}) := \{\omega \in \mathcal{A}^1_{S^1}(\mathbb{R}) \mid \oint \omega = 0\}$, we have $\mathcal{A}^0_{S^1,c}(\mathbb{R}) = d\mathcal{A}^0_{S^1}(\mathbb{R})$. (Ker($\oint$) $= d\mathcal{A}^0_{S^1}(\mathbb{R})$.)

Proof: For $F \in \mathcal{C}^\omega(\mathbb{R}, \mathbb{R})$ such that $\omega = dF$, the condition $\oint \omega = F(2\pi) - F(0) = 0$ means that $F \in \mathcal{C}^\omega(S^1, \mathbb{R}) \equiv \mathcal{A}^0_{S^1}(\mathbb{R})$. ■

Proposition

$\partial_t \mathcal{E}$ vanishes iff $k\partial_t k ds \in d\mathcal{A}^0_{S^1}(\mathbb{R})$ i.e., $\exists f \in \mathcal{A}^0_{S^1}(\mathbb{R})$ such that

$$k \boxed{\partial_t k} ds = \boxed{\partial_s f} ds \in d\mathcal{A}^0_{S^1}(\mathbb{R}).$$

Proof: $\partial_t \mathcal{E} = \partial_t \oint k^2 ds = \oint k\partial_t k ds = 0$ means $\oint \partial_s f ds = 0$ due to Lemma. ■

Infinitesimal Isometric & Isoenergy Diffeo. $\mathfrak{ii\partial iff}$ IV:

Key Facts for $\mathfrak{ii\partial iff}$

Key Fact 1: A trivial deformation $\partial_{t_1} k = \partial_s k$.

Key Fact 2: $\mathfrak{i\partial iff}$: $\partial_t k = \Omega_i U_i$, $\boxed{\partial_s U_r = k U_i}$

Key Fact 2:

$k\boxed{\partial_t k} = \boxed{\partial_s f}$ in Proposition recalls $k\boxed{U_i'} = \boxed{\partial_s U_r'}$.

Infinitesimal Isometric & Isoenergy Diffeo. $\mathfrak{iiDiff}$ V:

Key Fact 3: Proposition

The isometric diffeomorphisms with $t, t' \in (-\varepsilon, \varepsilon)$,

$$\partial_t Z = \ell(U_i)\partial_s Z = (U_r + \sqrt{-1}U_i)\partial_s Z, \quad (\partial_t k = \Omega_i U_i)$$

$$\partial_{t'} Z = \ell(U_i')\partial_s Z = (U_r' + \sqrt{-1}U_i')\partial_s Z, \quad (\partial_{t'} k = \Omega_i U_i')$$

with $\ell_d : \mathcal{A}^0_{S^1}(\mathbb{R}) \to d\mathcal{A}^0_{S^1}(\mathbb{R})$, $\boxed{kU_i' = \partial_s U_r'}$ and $kU_i = \partial_s U_r$.

If $\boxed{\partial_t k = U_i'}$, the energy $\boxed{\partial_t \mathcal{E} \text{ vanishes}}$, $\partial_t \mathcal{E} = \oint ds\ k\partial_t k = 0$.

Proof: $\partial_t \mathcal{E} = 2\oint ds\ k\partial_t k = 2\oint ds\ kU_i' = 2\oint ds\ \partial_s U_r' = 0$ ■

$$\boxed{\partial_t k = \Omega_i U_i.} \quad \text{and} \quad \boxed{\partial_{t'} k = \Omega_i U_i' = \Omega_i \partial_t k = \Omega_i^2 U_i}$$

Infinitesimal Isometric & Isoenergy Diffeo. $\mathfrak{iidiff}$ VI:

Key Fact 3: Corollary

The infinitesimal isometric diffeomorphism $\partial_t Z = \ell(U_i)\partial_s Z$ is **isoenergy**, **if** there exists another infinitesimal isometric diffeomorphism t', $\partial_{t'} Z = \ell(\partial_t k)\partial_s Z$, i.e.,

$$\partial_{t'} k = \Omega_i \partial_t k = \Omega_i^2 U_i.$$

Ascendant relations:

Isometric t is **isoenergy!** $\leftarrow$ $\exists$isometric t':$\partial_{t'} k = \Omega_i \partial_t k$

Isometric t' is **isoenergy!** $\leftarrow$ $\exists$isometric t'':$\partial_{t''} k = \Omega_i \partial_{t'} k$.

Isometric t'' is **isoenergy!** $\leftarrow$ $\exists$isometric t''':$\partial_{t'''} k = \Omega_i \partial_{t''} k$.

$\vdots$

Infinitesimal Isometric & Isoenergy Diffeo. $\mathfrak{iidiff}$ VII:

Summary of Key Facts

Key Fact 1: A trivial deformation is Isometric and isoenergy! $\partial_t Z = (1+0\sqrt{-1})\partial_s Z$, $\partial_t k = \partial_s k$,

$$\oint k\partial_t k ds = \oint \frac{1}{2}\partial_s k^2 ds = 0$$

Key Fact 3: $\mathfrak{iidiff}$ **is given as a sequence of** $\mathfrak{idiff}$**.**

Isometric t **is isoenergy!** $\leftarrow \exists$isometric t':$\partial_{t'} k = \Omega_i \partial_t k$

Remark

The trivial deformation $\partial_t k = \partial_s k$ **should be regarded that there might exist a flow of** t' **for**

$$\partial_{t'} k = \Omega\partial_t k = \Omega\partial_s k, \quad (U_i' = \partial_t k, U_r' = \frac{1}{2}k^2),$$

$$\partial_{t'} Z = \left(\frac{1}{2}k^2 + \sqrt{-1}\partial_s k\right)\partial_s Z = -\overline{\{Z, s\}_{\mathsf{SD}}}\partial_s Z.$$

Key Fact 4: Lemma

The deformation

$$\partial_{t'} k = \Omega \partial_s k \quad (U_i' = \partial_s k, U_r' = \frac{1}{2}k^2),$$

$$\partial_{t'} Z = \left(\frac{1}{2}k^2 + \sqrt{-1}\partial_s k\right)\partial_s Z = -\overline{\{Z, s\}_{\mathrm{SD}}}\partial_s Z,$$

is isoenergy.

Proof:

$$\begin{aligned}\partial_{t'}\mathcal{E} &= 2\oint k\partial_{t'} k \ ds \\ &= \oint (k^3 + 2\partial_s^2 k)\partial_s k ds \\ &= \oint \partial_s(\frac{1}{4}k^4 + (\partial_s k)^2) ds. \blacksquare\end{aligned}$$

Infinitesimal Isometric & Isoenergy Diffeo. $\mathfrak{iidiff}$ IX:

Ascendant infinitesimal isometric & isoenergy relations from the trivial relation

$$\partial_{t_1} k = \partial_s k. \qquad \mathfrak{so}(2)$$

$$\partial_{t_2} k = \Omega_i \partial_{t_1} k = \Omega_i \partial_s k, \qquad \Lambda\mathfrak{so}(2)$$

$$\partial_{t_3} k = \Omega_i \partial_{t_2} k = \Omega_i^2 \partial_{t_1} k = \Omega_i^2 \partial_s k, \qquad \Lambda\mathfrak{so}(2)$$

$$\partial_{t_4} k = \Omega_i \partial_{t_2} k = \Omega_i^2 \partial_{t_2} k = \Omega_i^3 \partial_{t_1} k = \Omega_i^3 \partial_s k, \qquad \Lambda\mathfrak{so}(2)$$

$$\vdots \qquad \qquad \vdots$$

Infinitesimal Isometric & Isoenergy Diffeo. $\mathfrak{iidiff}$ X:
The MKdV hierarchy

$\partial_{t_\ell} k = \Omega_i^{\ell-1} \partial_s k$ $(\ell = 1, 2, \ldots,)$ is the MKdV hierarchy:

$$\partial_{t_1} k - \partial_s k = 0,$$
$$\partial_{t_2} k - \frac{3}{2} k^2 \partial_s k - \partial_s^3 k = 0,$$
$$\partial_{t_3} k - 10 k \partial_s k \partial_s^2 k - \frac{5}{2} k^2 \partial_s^3 k - \frac{5}{2} (\partial_s k)^3 - \frac{15}{8} k^4 \partial_s k - \partial_s^5 k = 0,$$

The MKdV hierarchy $\partial_{t_\ell} k = \Omega_i^{\ell-1} \partial_s k$ $(\ell = 1, 2, \ldots,)$ are **isometric and isoenergy.**

Infinitesimal Isometric & Isoenergy Diffeo. $\mathfrak{iidiff}$ XI:

Fact 5. Lemma (Finite Dimension Condition)

If for $\ell \geq g+1$, $\partial_{t_\ell} k = \Omega_i^{\ell-1} \partial_s k \equiv 0$ or t_ℓ does not give a deformation essentially, t_ℓ is also a **trivial** isometric & isoenergy diffeomorphism.

Proof: $\partial_t \mathcal{E} = \partial_t \oint k^2 ds = 2 \oint k \partial_t k ds = 0$ ■

This makes $dZ = \partial_{t_2} Z dt_2 + \partial_{t_3} Z dt_3 + \cdots + \partial_{t_g} Z dt_g$: finite dimension;

$$\dim_{\mathbb{R}} T_Z \mathcal{M}^{\mathbb{C}}_{\mathrm{elas},E} = g - 1.$$

Infinitesimal Isometric & Isoenergy Diffeo. $\mathfrak{ii\partial iff}$ XII:

Summary of Key Facts

Key Fact 1: Trivial deformation $\partial_{t_1} k = \partial_s k$,

Key Fact 2: $\mathfrak{i\partial iff}$**:** $\partial_t k = \Omega_i U_i, \;\; \Omega_i := \partial_s(\partial_s + k\partial_s^{-1} k)$.

Key Fact 3: $\mathfrak{ii\partial iff}$ **is given as a sequence of** $\mathfrak{i\partial iff}$**.**

Isometric t is **isoenergy!** $\leftarrow$ $\exists$isometric t':$\partial_{t'} k = \Omega_i \partial_t k$

Key Fact 4: $\partial_{t_2} k = \Omega_i \partial_s k$.

Key Fact 5: Finite Dimension Condition $\partial_{t_\ell} k = 0$ for $\ell > g$.

Infinitesimal Isometric & Isoenergy Diffeo. $\mathfrak{iidiff}$ XIII:

Ascendant relations from the trivial relation

$\partial_{t_1} k = \partial_s k.$	$\mathfrak{so}(2)$
$\partial_{t_2} k = \Omega_i \partial_{t_1} k = \Omega_i \partial_s k,$	$\Lambda\mathfrak{so}(2)$
$\partial_{t_3} k = \Omega_i \partial_{t_2} k = \Omega_i^2 \partial_{t_1} k = \Omega_i^2 \partial_s k,$	$\Lambda\mathfrak{so}(2)$
$\partial_{t_4} k = \Omega_i \partial_{t_2} k = \Omega_i^2 \partial_{t_2} k = \Omega_i^3 \partial_{t_1} k = \Omega_i^3 \partial_s k,$	$\Lambda\mathfrak{so}(2)$
$\vdots$	$\vdots$
$\partial_{t_{g+1}} k = 0$	trivial
$\partial_{t_{g+2}} k = 0$	trivial

Infinitesimal Isometric & Isoenergy Diffeo. $\mathfrak{iidiff}$ XIV:
The MKdV hierarchy

$\partial_{t_\ell} k = \Omega_i^{\ell-1} \partial_s k$ $(\ell = 1, 2, \ldots, g)$ is the MKdV hierarchy:

$$\partial_{t_1} k - \partial_s k = 0,$$
$$\partial_{t_2} k - \frac{3}{2} k^2 \partial_s k - \partial_s^3 k = 0,$$
$$\partial_{t_3} k - 10 k \partial_s k \partial_s^2 k - \frac{5}{2} k^2 \partial_s^3 k - \frac{5}{2} (\partial_s k)^3 - \frac{15}{8} k^4 \partial_s k - \partial_s^5 k = 0,$$
$$\vdots$$
$$\Omega_i^g \partial_s k = 0$$

Infinitesimal Isometric & Isoenergy Diffeo. $\mathfrak{iidiff}$ XV:

Key Fact 1: Trivial deformation

1. There is a **trivial** isometric & isoenergy diffeomorphism,

$$\partial_{t_1} k = \partial_s k, \quad \mathfrak{so}(2)$$

Key Fact 1': Lemma

By considering $\partial_s^{-1} 0 = c \in \mathbb{R}$ and by letting $c = 1$, we have

$$\partial_s k = \Omega_i 0 = (\partial_s^2 + \partial_s k \partial_s^{-1} k) 0,$$

and thus

$$\partial_{t_1} k = \Omega_i 0.$$

Infinitesimal Isometric & Isoenergy Diffeo. $\mathfrak{iidiff}$ XVI:

$$\begin{pmatrix} 0 \\ \partial_{t_1} k \\ \partial_{t_2} k \\ \vdots \\ \partial_{t_g} k \end{pmatrix} = \begin{pmatrix} & & & & \Omega_i \\ \Omega_i & & & & \\ & \Omega_i & & & \\ & & \ddots & & \\ & & & \Omega_i & \end{pmatrix} \begin{pmatrix} 0 \\ \partial_{t_1} k \\ \partial_{t_2} k \\ \vdots \\ \partial_{t_g} k \end{pmatrix}.$$

This induces

$$dZ = \partial_1 Z dt_1 + \partial_2 Z dt_2 + \partial_3 Z dt_3 + \cdots + \partial_g Z dt_g,$$

and $\mathfrak{so}(2)^{(g+1)\oplus}$-actions ($\ell = 0, 1, \cdots, g$),

$$\begin{pmatrix} \partial_s & -k \\ -\partial_s k^{-1} \partial_s & \partial_s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0, \quad \begin{pmatrix} \partial_s & -k \\ -\partial_s k^{-1} \partial_s & \partial_s \end{pmatrix} \begin{pmatrix} \partial_s^{-1} k \Omega_i^\ell \partial_s k \\ \Omega_i^\ell \partial_s k \end{pmatrix} = 0.$$

(Adams-Harnad-Previato CMP 1988, Adler-Kostant-Symes system for loop algebra. (Previato-M 2014)

5. Isometric & Isoenergy Diffeo., IIDiff

(Pedit 1998, MO 2003, Fujioka-Kurose 2013)

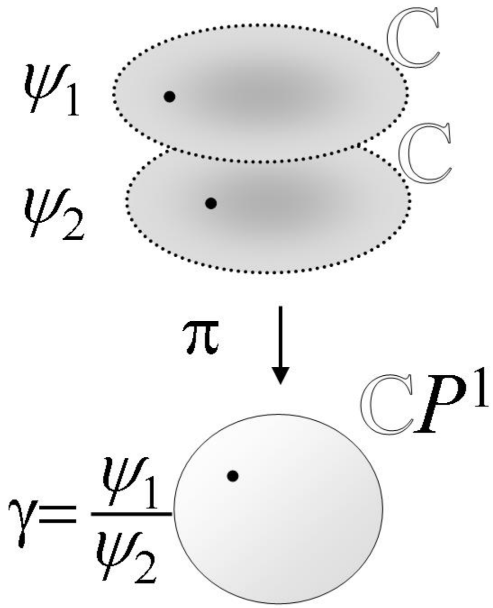
Isometric & Isoenergy Diffeo., IIDiff I
Quantized Elastica in $\mathbb{C}P^1$ I

$$\psi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1,$$
$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \mapsto \gamma := (\psi_1 : \psi_2)$$

For $\psi_2 \neq 0$, $\gamma = \dfrac{\psi_1}{\psi_2}$

$\mathrm{PSL}_2(\mathbb{C})$ acts on $\mathbb{C}P^1$ by

$$g_m : \gamma \mapsto \frac{a\gamma + b}{c\gamma + d}.$$



Isometric & Isoenergy Diffeo., IIDiff II

Quantized Elastica in $\mathbb{C}P^1$ II

$\mathcal{M}_{\text{elas}}^{\mathbb{C}^2\setminus\{0\}} := \{\psi : S^1 \to \mathbb{C}^2 \setminus \{0\}$
$\mid \det(\psi, \psi_s) = 1\}.$

$\leftrightarrow$

$\mathcal{M}_{\text{elas}}^{\mathbb{C}P^1} := \{\gamma : S^1 \to \mathbb{C}P^1$
$\mid |\partial_s\gamma| = 1\}.$

$\mathbb{C}^2 \setminus \{0\} \ni \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ up to $\mathrm{SL}_2(\mathbb{C})$.

$\longrightarrow$

$\mathbb{C}P^1 \ni \gamma = (\psi_1 : \psi_2)$ up to $\mathrm{PSL}_2(\mathbb{C})$.

Solution $\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ up $\mathrm{SL}_2(\mathbb{C})$ of

$$\left(-\partial_s^2 - \frac{1}{2}\{\gamma, s\}_{\mathrm{SD}}\right)\psi = 0,$$

$\det(\psi, \psi_s) = 1.$

$\longleftarrow$

$\{\gamma, s\}_{\mathrm{SD}}$ up to $\mathrm{PSL}_2(\mathbb{C})$.

$$\psi_\gamma := \begin{pmatrix}\sqrt{-1}\gamma/\sqrt{\partial_z\gamma}\\ \sqrt{-1}/\sqrt{\partial_z\gamma}\end{pmatrix}$$

with $\det(\psi_\gamma, \partial_z\psi_\gamma) = 1.$

Isometric & Isoenergy Diffeo., IIDiff III

Quantized Elastica in $\mathbb{C}P^1$ III

Isometric Deformation

The arc-length preserving deformation:

$$\gamma_t : S^1 \times (-\epsilon, \epsilon) \to \mathbb{C}P^1$$

such that by letting $\partial_t := \partial / \partial t$,

$$(\partial_t \partial_s - \partial_s \partial_t)\gamma_t = [\partial_t, \partial_s]\gamma_t = 0.$$

Isometric & Isoenergy Diffeo., IIDiff IV

Quantized Elastica in $\mathbb{C}P^1$ IV

Lemma

The arc-length preserving deformation is given by

$$\partial_t \psi_t = (A(s,t) + B(s,t)\partial_s)\psi_t,$$

for functions $A(s,t)$ and $B(s,t)$ of $\mathcal{C}^\infty(S^1 \times [0,1], \mathbb{C})$ such that $\partial_s B(s,t) = -2A(s,t)$.

Lemma

For the arc-length preserving deformation, $u := \{\gamma_{\mathbb{R}}, s\}_{\mathrm{SD}}/2$ satisfies

$$\partial_t u = -\Omega_{\mathbb{C}P^1} A$$

where

$$\Omega_{\mathbb{C}P^1}\partial_s = (\partial_s^3 + 2u\partial_s + 2\partial_s u).$$

Isometric & Isoenergy Diffeo., IIDiff V

Quantized Elastica in $\mathbb{C}P^1$ V

$$\partial_{t_n} u = \frac{1}{2}\Omega^n_{\mathbb{C}P^1}\partial_s u, \quad (n = 0, 1, \dots,) \tag{2}$$

$[\partial_{t_i}, \partial_{t_j}]\gamma_t(s) = 0$ $(i, j = 0, \dots,)$ is **Korteweg-de Vries(KdV) hierarchy** which preserves the energy:

$$\begin{aligned}
n = 0: &\quad \partial_{t_0} u + \partial_s u = 0, \\
n = 1: &\quad \partial_{t_1} u + 6u\partial_s u + \partial_s^3 u = 0, \\
n = 2: &\quad \partial_{t_2} u + 30u^2\partial_s u + 20\partial_s u\partial_s^2 u \\
&\qquad + 10u\partial_s^3 u + \partial_s^5 u = 0.
\end{aligned}$$

Isometric & Isoenergy Diffeomorphism, IIDiff VI: **MKdV and KdV Hierarchies**

The MKdV hierarchy $\partial_{t_\ell} k = \Omega_i^{\ell-1}\partial_s k$ ($\ell = 1, 2, \dots,$) gives **an isometric and isoenergy diffeomorphism** IIDiff due to the integrability. **(KdV case: MO 2003, Fujioka-Kurose 2014)**

Finite type solutions of the KdV and MKdV hierarchies are given by the **hyperelliptic functions! (Its, Matveev, Dubrovin, Novikov, Krichever, Date, Tanaka, $\cdots$)**

Isometric & Isoenergy Diffeomorphism, IIDiff VII:
The MKdV hierarchy

Theorem

The finite solution of the MKdV hierarchy is linearized in hyperelliptic Jacobian $\mathcal{J}_g = \mathbb{C}^g/\Gamma$ where Γ is a certain lattice $\mathbb{Z}^{2g}$, i.e., its orbit $\mathcal{O}$ agrees with $\mathcal{J}_g$.

Proposition (2003 MO, 2001,2002 M)

For a point $Z \in \mathcal{M}^{\mathbb{C}}_{\text{elas}}$ whose orbit $\mathcal{O}_Z$ of MKdV flow satisfies "finite type" condition, $\partial_{t_{j+1}} k = \Omega^j_i \partial_s k \equiv 0$, for $j \geq g$, the orbit $\mathcal{O}_Z$ is homeomorphic to

$$\mathcal{O}_Z \sim \mathbb{T}^\ell / S^1 \text{ for } \ell \leq g \text{ and } \mathbb{T}^\ell \subset \mathcal{J}_g,$$

where $\mathbb{T}^g$ is a real torus $\mathbb{T}^g := \prod^g S^1$.

Isometric & Isoenergy Diffeomorphism, IIDiff VIII: **The MKdV hierarchy**

Proposition (2003 MO, 2001,2002 M)

For a point $Z \in \mathcal{M}^{\mathbb{C}}_{\text{elas}}$ whose orbit $\mathcal{O}_Z$ of MKdV flow satisfies "finite type" equations,

$$\partial_{t_{j+1}} k = \Omega_i^j \partial_s k \equiv 0, \qquad \textbf{for } j \geq g,$$

the orbit $\mathcal{O}_Z$ is isometric & isoenergy; for every $Z' \in \mathcal{O}_Z$,

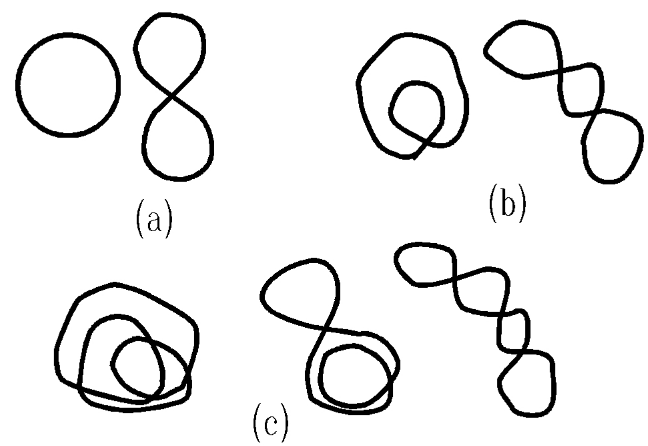
$$\mathcal{E}[Z'] = \mathcal{E}[Z], \ \textbf{or } Z' \in \mathcal{M}^{\mathbb{C}}_{\text{elas},\mathcal{E}[Z]}.$$

Definition

$$\mathcal{M}^{\mathbb{C}}_{\text{elas},g} := \bigcup_{\mathcal{O}_Z \sim \mathbb{T}^\ell/S^1, 1 \leq \ell \leq g} \mathcal{O}_Z \quad \subset \mathcal{M}^{\mathbb{C}}_{\text{elas}}.$$

Isometric & Isoenergy Diffeomorphism, IIDiff IX:
The MKdV hierarchy

1) The solution space contains **Euler's results** as genus one.
2) The solution of MKdV hierarchy is given by the **hyperelliptic curves including ∞ genus**.



Isometric & Isoenergy Diffeomorphism, IIDiff X:

The MKdV hierarchy

Theorem (2003 MO) : Filtration and Inductive Limit

$\mathcal{M}^{\mathbb{C}}_{\text{elas}}$ **has a filter structure and is given as the inductive limit of finite solution spaces of isometric & isoenergy deformation.**

$$\mathcal{M}^{\mathbb{C}}_{\text{elas}} = \lim_{\rightarrow} \mathcal{M}^{\mathbb{C}}_{\text{elas},g}, \quad \mathcal{M}^{\mathbb{C}}_{\text{elas},g} \subset \quad \mathcal{M}^{\mathbb{C}}_{\text{elas},g+1},$$

Proof: Note that the MKdV hierarchy is an initial problem. For every $Z \in \mathcal{M}^{\mathbb{C}}_{\text{elas}}$, there is an orbit $\mathcal{O}_Z$ of the MKdV hierarchy such that $Z \in \mathcal{O}_Z$ which is the inductive limit of $\mathcal{M}^{\mathbb{C}}_{\text{elas},g}$. ■

Isometric & Isoenergy Diffeomorphism, IIDiff XI:

The MKdV hierarchy

Theorem (Isometric & Isoenergy Diffeomorphism)

In $\mathcal{M}^{\mathbb{C}}_{\text{elas}}$, the spectrum decomposition

$$\mathcal{M}^{\mathbb{C}}_{\text{elas}} = \coprod_{E} \mathcal{M}^{\mathbb{C}}_{\text{elas},E},$$

and genus filtration induce the decomposition

$$\mathcal{M}^{\mathbb{C}}_{\text{elas}} = \coprod_{E} \bigcup_{g} \mathcal{M}^{\mathbb{C}}_{\text{elas},E,g}, \quad \mathcal{M}^{\mathbb{C}}_{\text{elas},E,g} := \mathcal{M}^{\mathbb{C}}_{\text{elas},E} \bigcap \mathcal{M}^{\mathbb{C}}_{\text{elas},g}.$$

Isometric & Isoenergy Diffeomorphism, IIDiff XII: **The MKdV hierarchy**

The expression of the partition function

$$\mathcal{W}_{\mathrm{elas}}[\beta] = \sum_E \mathrm{Vol}(\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},E}) \exp(-\beta E)$$

formally means that **the problem to evaluate the partition function is reduced to**

1. **determination of the isoenergy flow (orbit), and**
2. **evaluation of the volume of the flow (orbit),** $\mathrm{Vol}(\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},E})$**.** **(**$(\log \mathrm{Vol}(\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},E}))/\beta$ **is entropy.)**

6. Topological Properties of Moduli of Quantized Elastica

Topological Properties of Moduli of Quantized Elastica I
The MKdV hierarchy

Lemma (Maclachlan)

The modulus space of conformal equivalence classes of compact Riemann surfaces of genus g is simply connected.

For $\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},g} \to \mathfrak{M}_{\mathrm{elas},g}, \quad (Z(\mathbb{T}^g) \mapsto pt)$, we have

$$\mathfrak{M}_{\mathrm{elas},g} \subset \mathfrak{M}_{\mathrm{hyp},g}, \quad \mathfrak{M}_{\mathrm{hyp},g} \sim pt.$$

Topological Properties of Moduli of Quantized Elastica II

The MKdV hierarchy

Lemma (MO 2003)

Due to the relations $\mathcal{M}^{\mathbb{C}}_{\text{elas},g} \setminus \mathcal{M}^{\mathbb{C}}_{\text{elas},g-1} \sim \mathbb{T}^{g-1}$ and

$$pt \hookrightarrow S^1 \hookrightarrow \mathbb{T}^2 \hookrightarrow \mathbb{T}^3 \hookrightarrow \mathbb{T}^4 \hookrightarrow \mathbb{T}^5 \hookrightarrow \cdots ,$$

we have

$$\mathcal{M}^{\mathbb{C}}_{\text{elas},1} \hookrightarrow \mathcal{M}^{\mathbb{C}}_{\text{elas},2} \hookrightarrow \mathcal{M}^{\mathbb{C}}_{\text{elas},3} \hookrightarrow \cdots .$$

Topological Results of Loop Space

Theorem (Bott-Tu)

The cohomology of the loop space ΩS^n over S^n is given by

$$\mathsf{H}^p(\Omega S^n, \mathbb{R}) = \mathbb{R}\delta_{p \text{ mod } (n-1),0}.$$

For $n = 2$ case, the ring structure is given by

$$\mathsf{H}^*(\Omega S^2, \mathbb{R}) = \mathbb{R}[x]/(x^2) \cdot \mathbb{R}[e],$$

where degree$(e) = 2$ and degree$(x) = 1$.

$$\mathsf{H}^*(\Omega S^2, \mathbb{R}) = \mathbb{R} + \mathbb{R}x + \mathbb{R}e + \mathbb{R}xe + \mathbb{R}e^2 + \mathbb{R}xe^2 + \cdots .$$

Topological Properties of Moduli of Quantized Elastica IV

Topological Results of Loop Space

Since $\mathcal{M}^{\mathbb{C}}_{\text{elas}}$ is topologically decomposed by genus, we have:

Theorem (MO 2003)

For the forgetful functor for : $Diff \to Top$, we have

$$\mathrm{H}^*(\Omega S^2, \mathbb{R}) = \mathrm{H}^*(\mathrm{for}(\mathcal{M}^{\mathbb{C}}_{\text{elas}}), \mathbb{R})$$

i.e., for $\mathrm{H}^*(\Omega S^2, \mathbb{R}) = \mathbb{R}[x]/(x^2) \cdot \mathbb{R}[e]$, $\mathrm{H}^*(\mathrm{for}(\mathcal{M}^{\mathbb{C}}_{\text{elas}}), \mathbb{R}) = \Lambda_{\mathbb{R}}[dt_1, \epsilon]$, where $\Lambda_{\mathbb{R}}[dt_1, \epsilon]$ is a ring generated by dt_1 and

$$\epsilon = dt_1 + dt_2 \wedge (dt_1 i_{\partial_1}) + dt_3 \wedge (dt_1 i_{\partial_1}) + \cdots$$

with the wedge product and the degree: degree$(dt_i) = 1$:

$$\mathrm{H}^*(\mathrm{for}(\mathcal{M}^{\mathbb{C}}_{\text{elas}}), \mathbb{R}) = \mathbb{R} + \mathbb{R}dt_1 + \mathbb{R}\epsilon + \mathbb{R}\epsilon dt_1 + \mathbb{R}\epsilon^2 + \mathbb{R}\epsilon^2 dt_1 + \cdots .$$

Topological Approach of Moduli space III

Proof:

Since $\epsilon \cdot 1 = dt_1$, and $\epsilon^{n-1} \cdot dt_1 = \epsilon^n \cdot 1 = dt_n \wedge dt_{n-1} \wedge \cdots \wedge dt_2 \wedge dt_1$, we have

$$\begin{aligned}\Lambda_{\mathbb{R}}[dt_1, \epsilon] &= \mathbb{R} + \mathbb{R}dt_1 + \mathbb{R}\epsilon + \mathbb{R}\epsilon dt_1 + \mathbb{R}\epsilon^2 + \mathbb{R}\epsilon^2 dt_1 + \cdots \\ &= \mathbb{R} + \mathbb{R}dt_1 + \mathbb{R}dt_1 \wedge dt_2 + \mathbb{R}dt_1 \wedge dt_2 \wedge dt_3 + \cdots .\end{aligned}$$

Due to the **Bäcklund transformation**, $\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}$ is topologically given as **a telescopic type space** related to these genera. Hence we have

$$\mathrm{H}^*(\mathrm{for}(\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}), \mathbb{R}) = \Lambda_{\mathbb{R}}[dt_1, \epsilon].$$

7. Euler's elastica (classical solution)

$$\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},1}$$

Euler's elastica (classical solution) I

1. (**Deformation** $\mathcal{M}^{\mathbb{C}}_{\text{elas},1}$)

$$\partial_{t_1} k = \partial_s k,$$
$$\partial_{t_2} k = \Omega_i \partial_s k = 0.$$

$$\partial_s (k \partial_s^{-1} k + \partial_s) \partial_s k = 0, \quad \partial_s \left(k \left(\frac{1}{2} k^2 + a \right) + \partial_s^2 k \right) = 0.$$

$$\boxed{\tfrac{1}{2} k^3 + ak + b + \partial_s^2 k = 0.}$$

$$\boxed{\tfrac{1}{4} k^4 + ak^2 + 2bk + c + (\partial_s k)^2 = 0.}$$

Euler's elastica (classical solution) II

2. (**Fluctuation**) $\int dsk^2 \to \int ds(k + \delta t \partial_t k)^2$

$$= \int ds(k + \delta t \Omega_i U_i)^2$$
$$= \int ds(k^2 + 2\delta t k \Omega_i U_i + \delta t^2 (\Omega_i U_i)^2)$$

3. **The classical equation (energy minimum)**: obeying

$$\frac{2\delta \int ds \delta t k \Omega_i U_i}{\delta U_i} = 0.$$

Euler's elastica (classical solution) III

4. Classical governing equation: $\frac{2\delta \oint ds \delta t k \Omega_i U_i}{\delta U_i} = 0$:

$$\frac{2\delta \oint ds \delta t k \Omega_i U_i}{\delta U_i} = \frac{2\delta \oint ds \delta t k (\partial_s^2 + \partial_s k \partial_s^{-1} k) U_i}{\delta U_i}$$
$$= \frac{2\delta \oint ds \delta t (\partial_s^2 k + \frac{1}{2} k^3 + ak) U_i}{\delta U_i}$$

$$\partial_s^2 k + \frac{1}{2} k^3 + ak = 0.$$

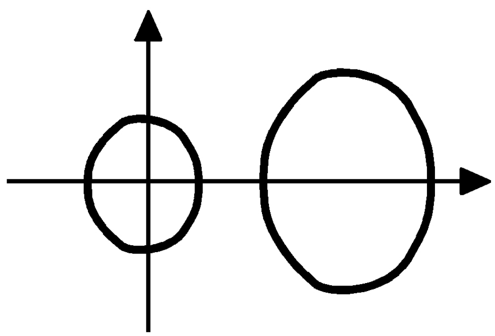
Euler's elastica (classical solution) IV

5. By integrating it and multiplying k, it becomes **SMKdV equation**

$$\partial_s(k^2) + 2\partial_s \frac{\partial_s^2 k}{k} = 0, \quad \to \quad \boxed{b + ak^2 + \tfrac{1}{4}k^4 + (\partial_s k)^2 = 0.}$$

$$(\partial_s k, k) \in \tilde{C}_1 := \left\{(\xi, \eta) | \xi^2 = -\frac{1}{4}\eta^4 - a\eta^2 - 4b.\right\}$$

Behind the problem, there exists the elliptic curve and elliptic integral:



$$s = \int^k \frac{dk}{\sqrt{-k^4/4 - ak^2 - 4b}}$$

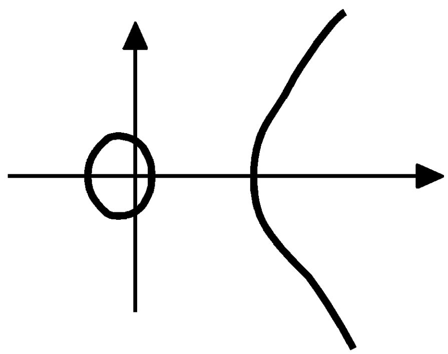
Euler's elastica (classical solution) V

6. (Another elliptic curve (2,3)) Let

$$x := \frac{1}{4}\sqrt{-1}\partial_s k + \frac{1}{8}k^2 + \frac{1}{4}a,$$
$$y := \frac{1}{2}\partial_s x = -\frac{1}{2}\left[\sqrt{-1}\left(-\frac{1}{8}k^3 - \frac{1}{4}ak + \frac{1}{4}\sqrt{-1}k\partial_s k\right)\right].$$

Then we have

$$y^2 = x\left(x - \frac{1}{4}a - \frac{1}{4}\sqrt{b}\right)\left(x - \frac{1}{4}a + \frac{1}{4}\sqrt{b}\right).$$



$$ds = \frac{dx}{2y}, \quad \sqrt{-1}k = \frac{\partial_s x}{x}, \quad \sqrt{-1}\phi = \log(x).$$

Euler's elastica (classical solution) VI

7. (an elliptic curve (2,3) and Weierstrass $\wp$ function)
   $\wp \equiv x - a/6$ where $\wp$ is the Weierstrass $\wp$ function.

8. (Euler's result from modern point of view)

$$\partial_s Z = \mathrm{e}^{\sqrt{-1}\phi} = x/\sqrt{-1} = (\wp(s) + a/6)/\sqrt{-1}.$$

   In other words, we have

$$\boxed{Z(s) = (-\zeta(s) + (a/6)s)/\sqrt{-1},}$$

9. The energy is given as $\boxed{\oint_{\alpha_1} k^2 ds = -4\eta' + 2(e_1)\omega'.}$

Euler's elastica (classical solution) VII

10. **The affine coordinate is proportional to the curvature, or the affine connection**.

$$X - X_0 = \frac{1}{4}k. \quad \text{:Euler's relation}$$

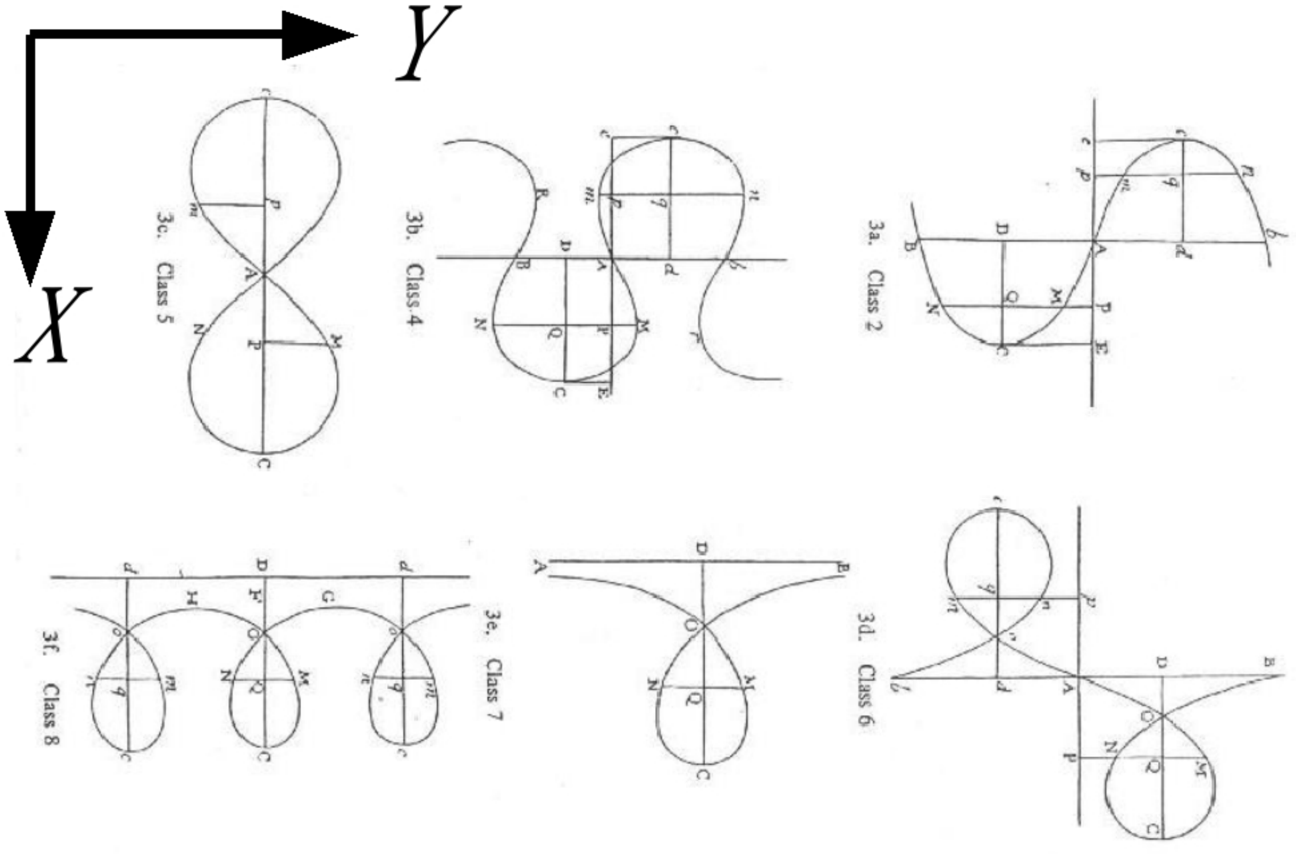
$$\wp(u - \omega') - \wp(u) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}u}\frac{\wp'(u)}{(\wp(u) - e_1)}.$$

11. (Euler's result from modern point of view)

$$s = \int^X \frac{\lambda^2 dX}{\sqrt{\lambda^4 - (\alpha + \beta X + \gamma X^2)^2}},$$

$$Y = \int^X \frac{(\alpha + \beta X + \gamma X^2) dX}{\sqrt{\lambda^4 - (\alpha + \beta X + \gamma X^2)^2}}.$$

Euler's elastica (classical solution) VIII



Isometric Diffeomorphism, IDiff XIII:

Euler's elastica (classical solution) VII

Moduli of Euler's elastica is defined by

$$\mathcal{M}^{\mathbb{C}}_{\text{Euler's elas}} := \{\tau : \text{Euler's elastica}\}$$

Though it is not closed, Euler implicitly found that

$$\dim \mathcal{M}^{\mathbb{C}}_{\text{Euler's elas}} = 1 \quad \text{and}$$

$$\mathcal{M}^{\mathbb{C}}_{\text{Euler's elas}} = \sqrt{-1}\mathbb{R}_{\geq 0} \cup (\frac{1}{2}\sqrt{-1}\mathbb{R}_{\geq 0}) \text{up to } \mathrm{SL}(2,\mathbb{C}).$$

Euler's elastica (classical solution) IX

11. **Euler's relation (from viewpoint of complex analysis)**

$$Z - Z_0 = \partial_s \log \partial_s Z.$$

$$\partial_u^2 \log \sigma(u) - \partial_u^2 \log \sigma(u)|_{u=\omega} = \frac{\mathrm{e}^{-\eta_1 z}\sigma(u-\omega_1)^2}{\sigma(u)^2}.$$

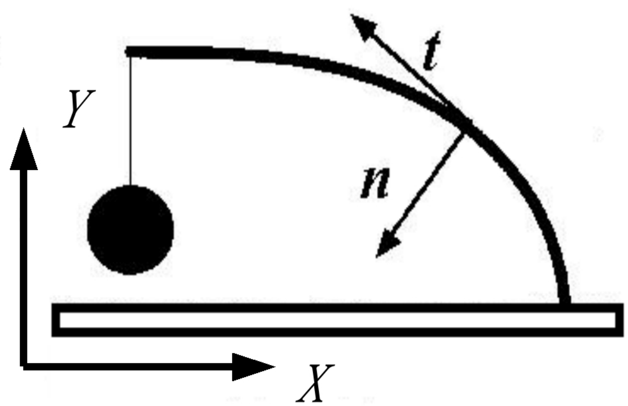
The map $\mathbb{C}^2 \setminus \{0\} \ni \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} \mapsto \gamma = \dfrac{\psi_1}{\psi_2} \in \mathbb{C}P^1$ induces

$$\mathbb{C}^2 \setminus \{0\} \ni \begin{pmatrix}\mathrm{e}^{-\eta_1 z}\sigma(u-\omega_1)^2 \\ \sigma(u)^2\end{pmatrix} \mapsto \partial_s\gamma = \frac{\mathrm{e}^{-\eta_1 z}\sigma(u-\omega_1)^2}{\sigma(u)^2} \in \mathbb{C}P^1.$$

Euler's elastica (classical solution) X

James Bernoulli found the Lemniscate integrals:

$$s = \int_X^1 \frac{dX}{\sqrt{1-X^4}}, \; Y = \int_X^1 \frac{X^2 dX}{\sqrt{1-X^4}}.$$



Euler found the **Legendre relation** of **the symplectic structure** in the Jacobian,

$$\int_0^1 \frac{dX}{\sqrt{1-X^4}} \int_0^1 \frac{X^2 dX}{\sqrt{1-X^4}} = \frac{\pi}{2}.$$

Euler's elastica (classical solution) XI
Symplectic Structure

$$\boxed{Z(s) = (-\zeta(s) + (a/6)s)/\sqrt{-1},}$$

The symplectic structure in Jacobian is given by

$$\langle ds, \zeta(s)ds\rangle = 1$$

and

$$\omega'\eta'' - \omega''\eta' = \frac{\pi}{2}\sqrt{-1}.$$

It means that for the space

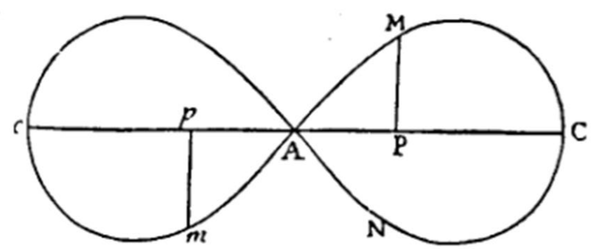
$$G := \{(s, Z(s))|s \in S^1\} \subset S^1 \times Z(S^1)$$

T_*G has the "symplectic structure" $ds \wedge dZ$.

Euler's elastica (classical solution) XII

12. Circle $X^2 + Y^2 = 1$.

13. Eight Figure:
The closed loops are only these cases. The shape of the Class 5 is realized by twisting of the circle. The modulus of the eight figure is $\tau = 0.70946\cdots \times \sqrt{-1}$, which corresponds to $\eta' = \omega' e_2/2$.



3c. Class 5

8. Quantized Elastica and Hyperelliptic Jacobians

Hyperelliptic Curves

In 1903, Baker gave the KP equation and KdV hierarchy using the bilinear operator and posed the problem similar to Novikov-conjecture starting from the theory of the hyperelliptic curves. H. F. Baker, On a system of differential equations leading to periodic functions, Acta Math. 27 (1903), 135156.

Since 1996, I have studied these hyperelliptic curves and algebraic curves using the Baker and Klein theory (2003 MO, 2001 M, 2002 M, 2007 EEMOP, 2008, EEMOP, 2008 MP, 2012KMP) with **Y. Ônishi, V. Enolskii, C. Eilbeck, Y. Kodama, J. Gibbons, and E. Previato.**

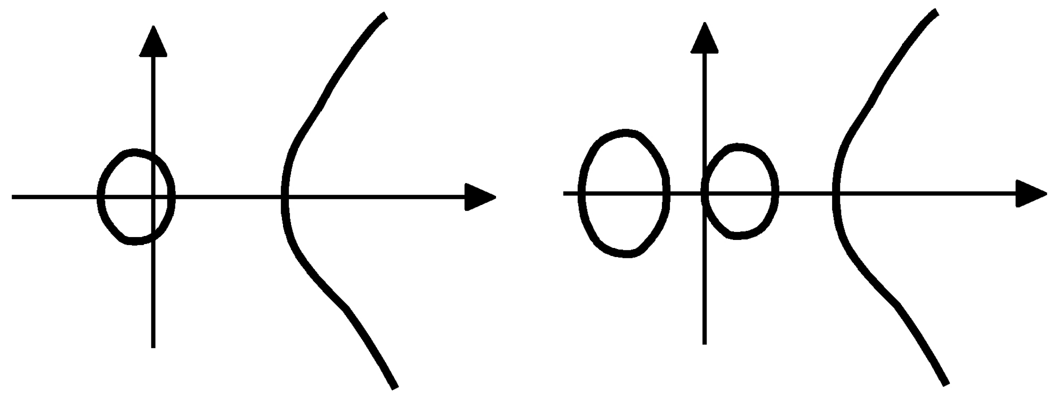
Quantized Elastica and Hyperelliptic Jacobians II

Hyperelliptic Curves

A hyperelliptic curve C_g of genus g ($g > 0$) is given by,

$$y^2 = (x - b_1)(x - b_2) \cdots \cdots (x - b_{2g+1}),$$

where b_j's are complex numbers.



$g = 1$ case
Euler's elastica

$g = 2$ case
Quantized elastica

Quantized Elastica and Hyperelliptic Jacobians II

Hyperelliptic Integrals

Hyperelliptic complete integrals :

$$\omega'_{ij} := \int_{\alpha_i} \nu^I_j, \quad \omega''_{ij} := \int_{\beta_i} \nu^I_j, \quad i, j = 1, \ldots, g,$$

$$\eta'_{ij} := \int_{\alpha_i} \nu^{II}_j, \quad \eta''_{ij} := \int_{\beta_i} \nu^{II}_j, \quad i, j = 1, \ldots, g,$$

where hyperelliptic differentials, 1st and 2nd kinds:

$$\nu^I_i = \frac{x^{i-1}dx}{2y}, \quad \nu^{II}_i = \frac{(x^{g+i-1} + \sum_{j=1}^{g+i-2} a_{ij}x^j)dx}{2y}.$$

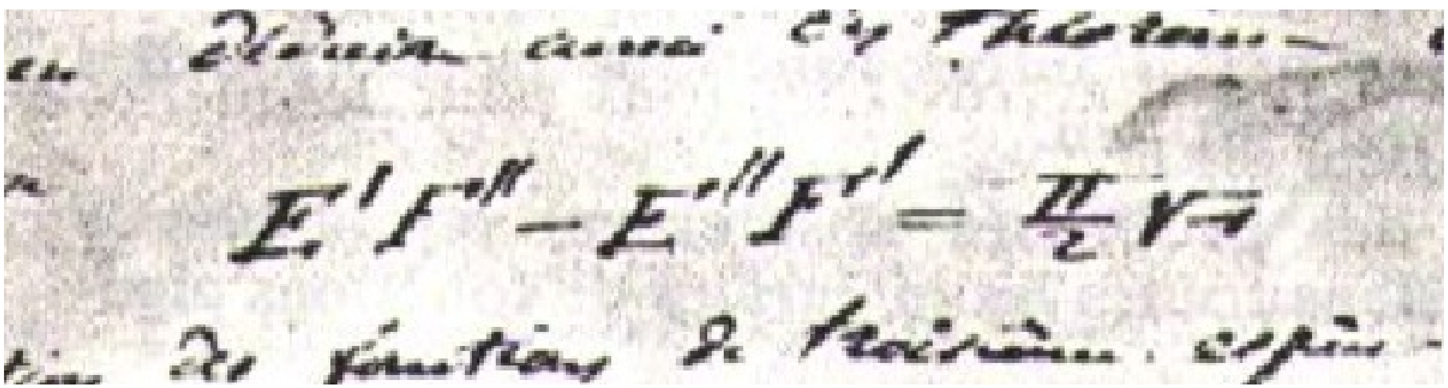
for certain a_{ij} of b_i's, $(i = 1, \ldots, g)$.

Symplectic structure as Legendre relations

Legendre relations as the symplectic structure:

$$\omega'\eta'' - \omega''\eta' = \frac{\pi}{2}\sqrt{-1}I_g$$

This is the same as a part of Galois's letter to A. Chevalier:


$$E'F'' - E''F' = \frac{\pi}{2}\sqrt{-1}$$

Hyperelliptic Jacobian

For a symmetric product space of C_g, $\mathsf{S}^g(C_g)$, the Abelian map is defined by

$$u := (u_1, \cdots, u_g) : \mathsf{S}^g(C_g) \longrightarrow \mathbb{C}^g,$$

$$\left(u_k((x_1, y_1), \cdots, (x_g, y_g)) := \sum_{i=1}^{g} \int_{\infty}^{(x_i, y_i)} \frac{x^{k-1} dx}{2y} \right).$$

The hyperelliptic Jacobian:

$$\mathcal{J}_g = \mathbb{C}^g / \Lambda, \quad \Lambda = < \omega', \omega'' >_{\mathbb{Z}} .$$

Quantized Elastica and Hyperelliptic Jacobians V
theta function and sigma function

$\mathbb{T} = \omega'^{-1}\omega''$. The θ function on $\mathbb{C}^g$ with modulus $\mathbb{T}$ and characteristics $\mathbb{T}a + b$ is given by

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z) = \theta\begin{bmatrix} a \\ b \end{bmatrix}(z;\mathbb{T})$$
$$= \sum_{n\in\mathbb{Z}^g} \exp\left[2\pi\sqrt{-1}\left\{\frac{1}{2}\ {}^t(n+a)\mathbb{T}(n+a) + {}^t(n+a)(z+b)\right\}\right]$$

for g-dimensional complex vectors a and b.

The σ-function is given by

$$\sigma(u) = \gamma_0 \exp\left\{-\frac{1}{2}\ {}^t u\eta'\omega'^{-1}u\right\}\vartheta\begin{bmatrix} \delta'' \\ \delta' \end{bmatrix}(\frac{1}{2}\omega'^{-1}u;\mathbb{T})$$

where δ and δ' are half-integer characteristics.

Quantized Elastica and Hyperelliptic Jacobians VI

Hyperelliptic $\wp$, ζ, and al_r functions

$$\wp_{ij} = -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u),$$

$$\zeta_i = \frac{\partial}{\partial u_i} \log \sigma(u)$$

$$\mathsf{al}_r := \sqrt{(b_r - x_1)(b_r - x_2)\cdots(b_r - x_g)} = \gamma_0' \frac{\mathrm{e}^{-\eta_r u} \sigma(u + \omega_r)}{\sigma(\omega_r)\sigma(u)},$$

Quantized Elastica and Hyperelliptic Jacobians VII

Hyperelliptic Solutions and Quantized Elastica

Theorem (2002, 2010 M)

1) For the hyperelliptic curve C_g**, by letting** $s := u_g$**,** $Z_r \in \mathcal{M}^{\mathbb{C}}_{\text{elas},E}$ $(r = 1, 2, \cdots, 2g+1)$ **is given by**

$$\partial_s Z_r(s) = \mathrm{al}_r(s)^2, \qquad Z_r(s) = b_r^g s - \sum_{i=1}^{g} \zeta_i(s) b_r^{i-1}.$$

2) $Z_r(u \in \mathcal{J}_g)$ **is isoenergy flows!!!**

3) The energy is given by the hyperelliptic integrals:

$$\oint_{\alpha_a} k_r^2 ds = -4\eta'_{ag} + 2(\lambda_{2g} + b_r)\omega'_{ag}$$

4) $\mathrm{Vol}(\mathcal{M}^{\mathbb{C}}_{\text{elas},E})$ **is the volume of the real subspace in the Jacobi variety** $\mathcal{J}_g$**.**

Quantized Elastica and Euler's Elastica

Remark

1) The shape of quantized elastica is

$$Z_r(s) = b_r^g s - \sum_{i=1}^g \zeta_i(s) b_r^{i-1},$$

whereas that of Euler's elastica is

$$Z(s) = (a/6)s - \zeta(s) \text{ for } (Z := Z(s)/\sqrt{-1}).$$

2) The energy of quantized elastica is

$$\oint k^2 ds = -4\eta'_{ag} + 2(\lambda_{2g} + b_r)\omega'_{ag},$$

whereas that of Euler's elastica is

$$\oint k^2 ds = -4\eta' + 2(e_1)\omega'.$$

3) The generalization of Euler's relation is

$$Z(u) - Z(u - \omega) = \sum_i^g b^{i-1} \partial_i \log \partial_{t_1} Z.$$

Quantized Elastica and Euler's Elastica

Remark

4) The shape of quantized elastica is

$$\begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_{g+1} \end{pmatrix} = \begin{pmatrix} b_1^g & b_1^{g-1} & b_1^{g-2} & \cdots & b_1 & 1 \\ b_2^g & b_2^{g-1} & b_2^{g-2} & \cdots & b_2 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{g+1}^g & b_{g+1}^{g-1} & b_{g+1}^{g-2} & \cdots & b_{g+1} & 1 \end{pmatrix} \begin{pmatrix} s \\ \zeta_g \\ \vdots \\ \zeta_1 \end{pmatrix}.$$

$\langle \zeta_r dt_{g-r}, dt_v \rangle = \delta_{r,v}$ **means** $\langle \sum_i \pi_{r,i} Z_i dt_{g-r}, dt_v \rangle = \delta_{r,v}$,

which is a "symplectic structure" in $\mathcal{M}^{\mathbb{C}}_{\text{elas}}$**.**

9. Final Remarks

9. Final Remarks

1. A quantized elastica in (p, q)-dimensional Minkswski space with so(p, q) and generalized MKdV equation.

2. **Willmore surface** (Polynakov extrinsic string) and MNV hierarchy (M 1999),

3. A geometrical object expressed by **generalized Weierstrass representation** of submanifold Dirac operator (M 2008, 2009),

4. **Diff/SDiff** for a manifold which B. Khesin (**Arnold-Khesin**) considers, or fluid dynamics.

Thanks!